HYDRODYNAMICS AND MASS TRANSFER IN A LIQUID LAYER AT A ROTATING SURFACE

N. S. Mochalova, L. P. Kholpanov, and V. Ya. Shkadov

UDC 532.526.75

The hydrodynamics and the mass transfer in a liquid layer at a rotating surface are analyzed in the boundary-layer approximation without undulation.

A centrifuge with a spiral channel rotating at a constant angular velocity [1-4] is a modified version of film-type heat exchangers with vortex flow. The thin liquid film in such an apparatus, while flowing from the center to the periphery along the inside surface, is in contact with a gas stream. The authors analyze here the hydrodynamics and the mass transfer in such an apparatus.

1. Let the x axis be an arc along the wetted wall of a spiral channel and let the y axis be normal to that wall. The liquid will be assumed incompressible, the flow will be assumed steady and isothermal. A thin layer of liquid moves without undulation along an Archimedes spiral whose equation in polar coordinates r, θ is $r = k\theta$ $(k > 0)$. It is assumed that the pressure gradient in the layer is due to rotation only. On these premises, then, the flow of a thin layer can be described by the Prandtl equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = X - \rho\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2},$$
$$-\frac{u^2}{R(x)} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{1}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

with the radius of curvature R(x) equal to

$$R(x) = \frac{(r^2 + r_\theta^2)^{3/2}}{r^2 + 2r_\theta^2 - rr_{\theta\theta}} = \frac{k(\theta^2+1)^{3/2}}{\theta^2+2}$$

in cylindrical coordinates, $r_\theta = dr/d\theta$, $r_{\theta\theta} = d^2r/d\theta^2$, and where X, Y are the projections of body forces on the x axis and the y axis, respectively. The body forces acting on the particles of a liquid film are the centrifugal force $F_c = \omega^2 R(x)$ and the Coriolis force of inertia $\overline{F_i} = 2\overline{\omega} \times \overline{v}$. If the spiral reverses its direction of rotation, this will be reflected only in the second of these forces. The projections of these body forces on the x and y axes are

$$X = \omega^2 R(x)\cos\alpha \pm 2\omega v,$$
$$Y = -\omega^2 R(x)\sin\alpha \mp 2\omega u, \tag{2}$$

where the upper sign corresponds to clockwise rotation and the lower sign corresponds to counterclockwise rotation, and where α is the angle formed by the vector of the centrifugal force and the tangent in the positive direction; since $\tan\alpha = r/r_\theta = \theta$, hence

$$\cos\alpha = \frac{1}{\sqrt{\theta^2+1}}, \quad \sin\alpha = \frac{\theta}{\sqrt{\theta^2+1}}.$$

M. V. Lomonosov Institute of Mechanics, Moscow State University. Institute of Petrochemical Processes, Academy of Sciences of the USSR. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 25, No. 4, pp. 648-655, October, 1973. Original article submitted March 4, 1973.

In the variables θ, y the system of equations (1) becomes

$$\frac{1}{k\sqrt{\theta^2+1}}u\frac{\partial u}{\partial \theta}+v\frac{\partial u}{\partial y}=X-\frac{1}{\rho}\frac{1}{k\sqrt{\theta^2+1}}\frac{\partial p}{\partial \theta}+\nu\frac{\partial^2 u}{\partial y^2},$$
$$-\frac{u^2}{R(\theta)}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y},\tag{3}$$
$$\frac{1}{k\sqrt{\theta^2+1}}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial y}=0.$$

From the condition of adhesion at the wall it follows that

$$u=0,\quad v=0\tag{4}$$

at y = 0.

We will solve system (3) by the method of integral relations. With friction at the surface assumed negligible, we have at $y = \delta$ the conditions

$$\frac{\partial u}{\partial y}=0,\ p=p_{\alpha}=\text{const},\ u=v.\tag{5}$$

The boundary conditions (4) and (5) are satisfied by the second-degree polynomial

$$u=v\left[2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2\right],\tag{6}$$

with v found from the expression for the flow rate $\int\limits_0^{\delta(x)} u\,dy = q = \text{const}$:

$$v=\frac{3}{2}\frac{q}{\delta}.\tag{7}$$

Eliminating the pressure from the second equation in (3), we find $\partial p/\partial\theta$. Using this quantity and integrating the first equation in (3) with respect to y over the entire boundary layer, we obtain an equation which is nonlinear with respect to $\delta(\theta)$. In dimensionless coordinates this equation becomes

$$\frac{d\bar{\delta}}{d\theta}=\frac{\dfrac{\bar{\omega}^2 k\bar{\delta}(\theta^2+1)}{\theta^2+2}-\dfrac{\bar{\omega}^2\bar{\delta}^2(\theta^4+5\theta^2+2)}{2\sqrt{\theta^2+1}(\theta^2+2)^2}-\dfrac{33}{40}\dfrac{\theta(\theta^2+4)}{k^2(\theta^2+1)^3}-\dfrac{3}{\mathrm{Re}\,\bar{\delta}^2}}{-\dfrac{6}{5}\dfrac{1}{\bar{\delta}^2 k\sqrt{\theta^2+1}}\mp\dfrac{\bar{\omega}}{k\sqrt{\theta^2+1}}+\dfrac{\bar{\omega}^2\theta\sqrt{\theta^2+1}}{\theta^2+2}\bar{\delta}},\tag{8}$$

where the upper sign corresponds to clockwise rotation and the lower sign corresponds to counterclockwise rotation.

Equation (8) was solved numerically, on a model M-20 computer, by the Runge–Kutta method with an automatic step adjustment. The curves in Fig. 1a represent a typical relation between the thickness of the liquid film and the length of the spiral on which it has formed, for various values of the hydrodynamic parameters.

The film thickness in a spiral centrifuge becomes constant under all known operating conditions. The time necessary for the film thickness to become constant is a function of the Reynolds number and of the angular velocity: as either increases, the apparatus will operate with a constant film thickness sooner. When θ is large, this constant film thickness can be found from Eq. (18) analytically by inserting there $d\bar{\delta}/d\theta = 0$:

$$\delta=\sqrt[3]{\frac{3\nu\theta_0 q}{r_0\omega^2}}.\tag{9}$$

According to (9), the film thickness increases with higher flow rates and decreases with higher angular velocity, depending also on the characteristic spiral parameter r_0/θ_0.

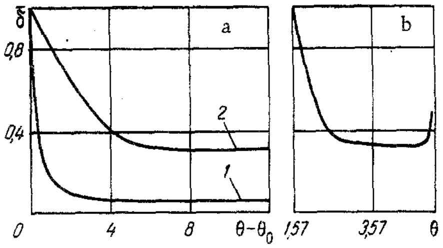

Fig. 1

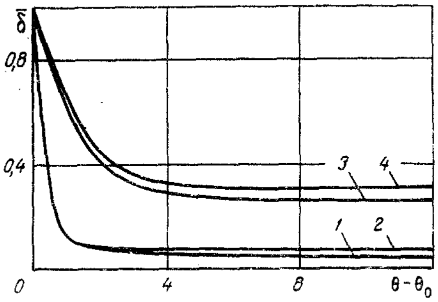

Fig. 2

Fig. 1. Dimensionless thickness of liquid film, as a function of the spiral length: **a:** for 1) $\bar{k} = 1$ and $\bar{\omega} = 10$ with Re = 100; 2) $\bar{k} = 1$ and $\bar{\omega} = 1$ with Re = 100; **b:** for $\bar{k} = 1$ and $\bar{\omega} = 1$ with Re = 20; $\theta \in [\pi/2, 2\pi]$.

Fig. 2. Thickness of the liquid film (curves 1 and 3) and thickness of the diffusion layer (difference between curves 1 and 2 or between curves 3 and 4), as functions of the spiral length: 1, 2) $\bar{k} = 1, \bar{\omega} = 10$, Re = 100, Pr = 1000; 3, 4) $\bar{k} = 1$, $\bar{\omega} = 1$, Re = 100, Pr = 100.

Under normal operating conditions the film spreads thinly over the walls of the spiral channel. For a solution corresponding to this condition it is required that $d\bar{\delta}/d\theta$ be negative. Otherwise the operation of a spiral centrifuge would break down and "stalling" would result, as indicated in Fig. 1b. The film thickness first decreases, as usually, but then increases rapidly at some distance along the spiral ($\theta \approx 5$), which means this mass exchanger has ceased to operate normally. The "stalling" condition is largely affected by the spiral parameter $\bar{k}$ and the angular velocity. We will, therefore, evaluate the "stalling" conditions at various speeds.

a. $\bar{\omega} \ll 1$. Then $d\bar{\delta}/d\theta$ is positive, i.e., at low speeds the apparatus almost always "stalls."

b. $\bar{\omega} \gg 1$. Now

$$\bar{k} < \frac{\bar{\delta}(\theta^4 + 5\theta^2 + 2)}{2(\theta^2 + 1)^{3/2}(\theta^2 + 2)}\,. \tag{10}$$

For a given spiral at high speeds it is possible to match h_0 and θ_0 so as to satisfy condition (10) and avoid "stalling."

c. At intermediate speeds the apparatus operates without "stalling," when

$$\frac{\bar{\delta}(\theta^4 + 5\theta^2 + 2)}{2(\theta^2 + 1)^{3/2}(\theta^2 + 2)} < \bar{k} < \frac{7(\theta^2 + 2)}{4\bar{\delta}\bar{\omega}\theta(\theta^2 + 1)}\,. \tag{11}$$

We will now determine the friction force τ on a liquid film along the entire length L of a spiral centrifuge:

$$\tau = \int_0^L \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} dx = 1.44\rho\nu^{1/3}\left(\frac{r_0}{\theta_0}\right)^{2/3} q^{1/3}\omega^{4/3}L. \tag{12}$$

2. If the resistance to mass transfer is concentrated within the liquid phase, then the coefficient of mass transfer to the liquid phase can be calculated from the equation of convective diffusion

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2}\,, \tag{13}$$

where velocities u and v must be taken from the solution to the hydrodynamic problem (6). It is assumed that the diffusivity remains constant and that the molecular transport is much smaller transversely than lengthwise. We now introduce a new system of coordinates by a following transformation of the original system: $y_1 = -y + \delta$ and $x_1 = x$, where the boundary conditions are

$$\text{for } y_1 = 0 \quad c = c_\infty,$$

$$\text{for } y_1 = \delta_d \quad \frac{\partial c}{\partial y} = 0, \; \frac{\partial^2 c}{\partial y^2} = 0, \; c = 0.$$

These conditions are satisfied by the third-degree polynomial

$$\frac{c(x, y)}{c_\infty} = \left(1 - \frac{y_1}{\delta_d}\right)^3. \tag{15}$$

Integrating (13) with respect to y from $y = \delta - \delta_d$ to $y = \delta$, i.e., over the thickness of the diffusion layer, and considering the boundary conditions (14), we obtain

$$\frac{ds^2}{dx} = \frac{80D}{(5 - s^2)\delta_d}, \tag{16}$$

where $s = \delta_d/\delta$.

The solution to Eq. (16) will be written in quadratures:

$$\delta_d = \delta \sqrt{5 - \sqrt{25 - \frac{160D}{q}\int \frac{dx}{\delta(x)}}} \tag{17}$$

For the flow mode with a constant film thickness Eq. (17) becomes

$$\delta_d = \delta \sqrt{5 - 5\sqrt{1 - \frac{ax}{25}}}, \tag{18}$$

where

$$a = \frac{160D}{q\delta}. \tag{19}$$

Let us now find the length of path along the spiral where the diffusion layer attains full growth, i.e., its thickness becomes equal to that of the liquid film.

From (18) we have the length through which the diffusion layer grows

$$L^* = \frac{9}{160}\frac{q\delta}{D}. \tag{20}$$

The value found for the thickness of the diffusion layer will be used for determining the mean coefficient of mass transfer to the liquid film. For this, the diffusion current to the interphase boundary will be averaged over some characteristic distance L

$$\beta = \frac{1}{LC_\infty}\int_0^L D\left(\frac{\partial c}{\partial y}\right)_{y=\delta} dx = 60\frac{D}{\delta a L}\left[5\left(1 - \sqrt{1 - \frac{aL}{25}}\right)^{1/2} - \frac{1}{3}\left(1 - \sqrt{1 - \frac{aL}{25}}\right)^{3/2}\right]. \tag{21}$$

The characteristic dimension of a spiral can vary, depending on the test conditions and depending on the actual geometrical length L of the spiral. For this reason, we distinguish several possible cases:

a. $L \ll L^*$, where L^* is calculated according to formula (20). In this case the thickness of the diffusion layer is much less than that of the liquid film, i.e., the diffusion layer does not build up to the film thickness. In physical terms, this corresponds to high values of the Prandtl number (Fig. 2). The length of the spiral L becomes its characteristic dimension. Then, expanding the expression under the square-root sign in (21) into a series and retaining the first term only, we obtain a formula for the coefficient of mass transfer to the liquid phase

$$\beta = \frac{60}{\sqrt{2}}\frac{D}{\delta L^{1/2} a^{1/2}}\left[1 - \frac{1}{750} aL\right],$$

or, taking into account (9) and (19),

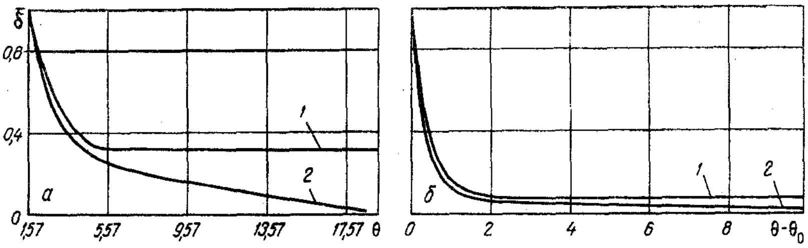

Fig. 3. Thickness of the liquid film (curve 1) and thickness of the diffusion layer (difference between curves 1 and 2), as functions of the spiral length: a) for $\bar{k} = 1$, $\bar{\omega} = 1$, Re = 100, Pr = 100, $\theta \in [\pi/2, 6\pi]$; b) for $\bar{k} = 1$, $\bar{\omega} = 1$, Re = 100, Pr = 200, $\theta \in [0, 6\pi]$.

$$\beta = 2.8 \left(\frac{r_0}{\theta_0}\right)^{1/6} \frac{D^{1/2}}{\nu^{1/6}} \frac{\omega^{1/3} q^{1/3}}{L^{1/2}} \left[1 - 0.147 \frac{DL\omega^{2/3}}{\nu^{1/3} q^{4/3}} \left(\frac{r_0}{\theta_0}\right)^{1/3}\right]. \tag{22}$$

We note that β is proportional to $D^{1/2}$.

b. $L \approx L^*$. The length of the spiral is now comparable with that of the fully grown diffusion layer. At distance L^* the thickness of the diffusion layer becomes comparable with that of the liquid film. In physical terms, this corresponds to relatively low values of the Prandtl number (Fig. 3). In this case the distance over which the diffusion layer attains full growth becomes the characteristic dimension. The mass-transfer coefficient is now determined by inserting the value of L^* from (20) into (21):

$$\beta = 14.7 \frac{D}{\delta}$$

or, taking into account (9),

$$\beta = 10.2 \frac{D}{\nu^{1/3}} \left(\frac{r_0}{\theta_0}\right)^{1/3} \frac{\omega^{2/3}}{q^{1/3}}. \tag{23}$$

In this case the mass-transfer coefficient is proportional to the diffusivity and an increase in the flow rate of the liquid reduces the mass transfer.

c. $L \gg L^*$. The length of the spiral exceeds by far the length of the diffusion layer. In this case, beginning at some distance, liquid will accumulate on the spiral wall. In order to account for this in the calculation of the mass-transfer coefficient, one must solve the same equation of convective diffusion (13) but with different boundary conditions:

$$\text{for } y = \delta \quad c = c_\infty \quad \text{(at the interphase boundary)}$$
$$\text{for } y = 0 \quad \frac{\partial c}{\partial y} = 0, \; c = c_{0(x)} \quad \text{(at the spiral wall).} \tag{24}$$

Moreover, the concentration at the spiral wall must be determined from that solution. The boundary conditions (24) are satisfied by the second-degree polynomial

$$\frac{c(x, y)}{c_\infty} = \frac{c_{0(x)}}{c_\infty} + \left(1 - \frac{c_{0(x)}}{c_\infty}\right) \frac{y^2}{\delta^2}. \tag{25}$$

Solving Eq. (13) with the boundary condition (25) by the method described here, we find an expression for the unknown concentration at the spiral wall:

$$\frac{c_{0(x)}}{c_\infty} = 1 - \exp\left(-\frac{40}{11\,\mathrm{Pe}} \int \frac{dx}{\delta(x)}\right). \tag{26}$$

The mass-transfer coefficients must, of course, be averaged over the distance from 0 to L (L denoting the length of the spiral). Moreover, (15) describes the concentration profile from 0 to L^* and (25) describes it from L^* to L.

According to the definition $\beta = \frac{1}{Lc_\infty}\int_0^L D\left(\frac{\partial c}{\partial y}\right)_{y=0} dx$, we find

$$\beta = 14.7\,\frac{D}{\delta} + \frac{11}{20}\,\frac{\mathrm{Pe}\,D}{L}\left[0.815-\exp\left(-\frac{40L}{11\,\mathrm{Pe}\,\delta}\right)\right], \tag{27}$$

where δ and L^* are given by formulas (9) and (19).

This analysis of possible mass-transfer modes leads to the following conclusion: the coefficient of mass transfer to the liquid phase is a function of the diffusivity and of the hydrodynamic parameters, but different for short spirals where the diffusion layer cannot fully grow (case a), for medium-long spirals where the thickness of the diffusion layer is comparable with the film thickness (case b), and for long spirals where a boundary layer builds up in the process of convective diffusion (case c).

NOTATION

u, v	are the projections of the velocity vector $\vec{V}$ on the x and y axes, respectively;
p	is the pressure in the liquid film;
R(x)	is the radius of curvature of the spiral;
X, Y	are the projections of body forces on the x and y axes, respectively.
p_a = const	is the pressure at the surface of the liquid layer;
h_0	is the initial film thickness;
$q = v_0 h_0$	is the flow rate of liquid;
$v = (3/2)(q/\delta)$	is the characteristic velocity;
$\bar{\delta} = \delta/h_0$	is the dimensionless film thickness;
$\bar{k} = k/h_0 = r_0/\theta_0 h_0$	is the dimensionless parameter of an Archimedes spiral;
$\omega = \bar{\omega}(v_0/h_0)$	is the angular velocity of a rotating spiral channel;
δ_d	is the thickness of the diffusion layer;
$\mathrm{Re} = q/\nu$	is the Reynolds number;
$\mathrm{Pe} = q/D$	is the Peclet number.

LITERATURE CITED

1. A. M. Arkharov, Trudy MVTU im. Baumana, No. 53 (1955).
2. V. L. Pebalk, Trudy MITKhT im. Lomonosova, No. 5 (1955).
3. S. Ya. Gershch and A. M. Arkharov, Izv. Vuzov, Mashinostroenie, No. 4 (1959).
4. N. A. Nikolaev, O. K. Ostrov, V. V. Shchukin, and F. A. Mustashkin, Trudy KKhTN im. Kirova, No. 37 (1968).